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T.B. Duishenaliev[1], duishenaliev@mail.ru
R.N. Askarbekov[2], askarbekovu@gmail.com

[1]Moscow Power Engineering Institute (MPEI)
[2]Kyrgyz State Technical University named after I.Razzakov,

MATHEMATICAL MODEL FOR DESCRIBING ELASTIC DEFORMATION OF RUBBER-METAL ELEMENTS

In this paper was proposed mathematical model for description of elastic deformation. The model is based on the new (non-classical) solution of the second boundary value problem of mechanics' deformable body. These solutions give us more suitable description of deformation in rubber layers in rubber-metal supports (RMS). We consider RMS under the influence of a distributed or concentrated load. The numerical value calculation gives us the diagram of stress-strain condition of the rubber in the composition of single-layer and multilayer of rubber.

Keywords: boundary value problem, strain tensor, stress tensor, displacement, load, rubber, rubber-metal support.

Introduction

Engineers and designers are using new materials in different industries, the number of new materials is growing every year. The properties of some of them need to be determined, measured anew. New software systems are used to simulate the behavior of these new materials under the influence of various loads. Of course, all engineers need to create a mathematical model to develop an algorithm for modeling in software systems. In classical theories, such as the theory of elasticity, there are various options for general solutions [8-10]. Where displacements and stresses are expressed to satisfy equality through arbitrary independent functions.

A general theory of the plane problem is constructed using the solution of boundary value problems with the help of analytical functions, which are based on the Cauchy integrals [1-4]. Also, mechanical scientists studied the possibility of constructing a solution to boundary value problems based on analytical functions for a spatial setting. In specific problems of the theory of elasticity, the solutions of Papkovich-Neuber and Galerkin [1-4] are used. Determining the stress-strain state in a rectangular parallelepiped under the action of external forces is an interesting problem, both from a practical and a theoretical point of view.

When solving specific problems, various methods are used today: potential theory, variational principles of group analysis, separation of Fourier variables.

General solutions of the equations of the theory of elasticity are an important tool for solving specific problems. Using the tensor of stress functions Krutkov [1, 4] obtained the solutions of Galerkin, Papkovich [1, 4] and he showed that it is possible to form a symmetric tensor $\gamma pq=\gamma qp$ of the second rank from the stress functions of Maxwell and Morera. The question of the general solution of the equilibrium equations using stress functions was considered in many works. The importance and significance of stress functions are discussed in a large review. Links to later works can be found in [4]. Some representations of the general solution of the equations of motion are found in [4]. In work [1, 4] Ostrosablin presented 17 equivalent forms of the general solution of the equilibrium equations of a continuous medium, i.e., 17 forms of stress representation in terms of three stress functions. Exhaustive versions of the general solution of the equations of motion of a continuous medium through 6 functions of stresses in displacements are obtained based on the approach [1, 4]. 17 forms [4] are obtained for the general solution of the equilibrium equations in the absence of time derivatives.

The solution of the Lamé equations was presented by Betty, Lichtenstein, Cerruti in terms of the harmonic vector and the radius vector xi. Deev solution [33] involves several more constants, which can be assigned different meanings. Grodsky, Papkovich, Neuber [4] represented displacements through four harmonic functions. Boussinesq and Galerkin [4] expressed displacements in terms of three biharmonic functions. Ostrosablin [4] proved the generality of the Papkovich-Neuber solution, as well as the Kelvin-Lame and Galerkin solutions. A connection is established between general solutions and symmetry operators, which are formulas to produce new solutions based on a specific solution.

The Saint-Venant compatibility conditions make it possible to find solutions for calculating the displacement under known strains and stresses (Maxwell / Morera), but now there are some uncertainties in the expression conditions themselves.

This topic is discussed in various literary sources [1, 4], the number of independent compatibility conditions and their modifications, the generality and completeness of stress functions, statement of the problem of the theory elasticity in stresses.

There are many works in which the formulation of boundary value problems in displacements / stresses is considered, that is, various versions of general solutions of the theory of elasticity are used. All existing works prove that the opinions of scientists still differ. When solving specific problems, the application of the classical theory becomes possible with the use of various additional conditions, additional components. Classical equations consist of equations of motion, generalized Hooke's law, Cauchy relations, etc. [5, 6, 11, 12]

Consider the first boundary value problem: the displacements of points of a material (elastic) body are known, which has a certain volume ***V*** and a boundary ***S***.

Let us consider the second boundary value problem: the external forces applied to a material (elastic) body are known, which has a certain volume ***V*** and a boundary ***S***.

1) $u_j = \bar{u}_j,\ x_i \in S$, 2) $\sigma_{ij} n_j = p_i,\ x_i \in S$

The stresses and displacements are continuous, and the function itself is sufficiently smooth along the entire boundary ***S*** of the body.

Having analyzed the existing solutions of boundary value problems, Professor Duishenaliev T.B. proposed a new unconventional method constructed in the coordinates of the final state (Euler coordinates) [1, 4, 7]. The essence of the method is that the linear Cauchy tensor describes the finite deformations of a material body, under certain conditions.

Let me give you a mathematical formulation of this problem using an unconventional method, please look at Figure1. Let *fi* and *Pi*, respectively, be the external forces given in the volume of the material body ***V*** and on the boundary of the body ***S***.

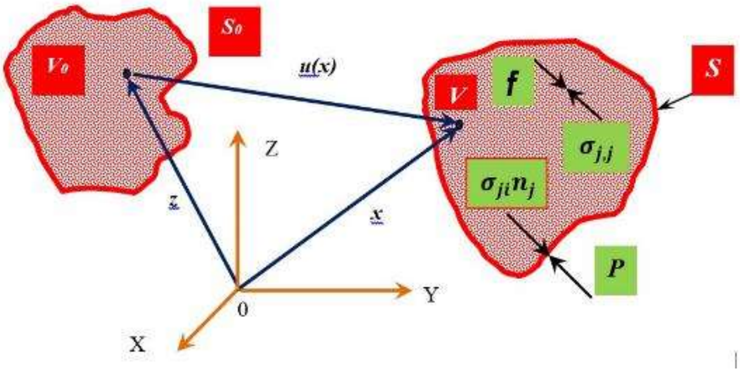

Fig. 1. *Left*: the state of equilibrium of the body without external forces,
right: the state of equilibrium of the body with the applied forces

$$\sigma_{ji,j}+f_i=0,\ \sigma_{ij}=\sigma_{ji},\ x_i\in V, \tag{1}$$

$$\sigma_{ij,kk}+\frac{1}{1+\nu}\sigma_{kk,ij}+\frac{\nu}{1-\nu}\delta_{ij}f_{k,k}+f_{i,j}+f_{j,i}=0,\ x_i\in V, \tag{2}$$

$$\sigma_{ji}n_j=p_i,\ x_i\in S, \tag{3}$$

where ν - Poisson's ratio, σ_{ij} - stress components, f_i and p_i respectively, external forces specified in the volume *V* and at the boundary *S* of the material body.

External forces are applied to each point of a material body of volume ***V*** and boundary ***S***, which are balanced by internal stresses. Functions $\sigma_{ij}(x)$ satisfying to the equations (1-3) may also be solution in spatial coordinates (coordinates of the final state) of this boundary value problem [4]. Deformations are easily determined from such a solution:

$$\varepsilon_{ij}=\frac{1}{E}\left(-\nu\cdot\delta_{ij}\cdot\sigma_{kk}+(1+\nu)\sigma_{ij}\right), \tag{4}$$

where ***E*** is the modulus of elasticity.

The displacements are determined by the Cesaro formulas [4]:

$$u_i(x)=u_i\left(x^0\right)+\omega_{ij}\left(x^0\right)\left(x_j-x_j^0\right)+\frac{1}{E}\int_l\left(\varepsilon_{ik}(y)+\left(x_j-y_j\right)\left(\varepsilon_{k,ij}(y)-\varepsilon_{k,ji}(y)\right)\right)dy_k$$

where ***l*** is a line in region ***V***, x^0 - starting point of this line, $u_i\left(x^0\right),\omega_{ij}\left(x^0\right)$ - integration constants.

Simulation of Stress-strain state of a single-layer rubber-metal supports

Let us set the domain of definition of the equations of the static boundary value problem in the form indicated in Fig. 2 rubber-metal elements with one rubber layer. We place the origin of the rectangular Cartesian coordinate system at the center, which corresponds to the position (X, Y, Z) = (0,0,0).

By the notation ***V*** we mean the following region (we use metric):

$$-0.05\le x_1\le 0.05,\ -0.05\le x_2\le 0.05,\ 0\le x_3\le 0.1 \tag{5}$$

Let us consider the second boundary value problem without mass forces:

$$\sigma_{ji,j}=0,\ \sigma_{ij}=\sigma_{ji},\ x_i\in V, \tag{6}$$

$$\sigma_{ij,kk}+\frac{1}{1+\nu}\sigma_{kk,ij}=0,\ x_i\in V, \tag{7}$$

$$\sigma_{ji}n_j=\delta_{i2}cx_3,\ x_i\in S, \tag{8}$$

where ***V*** is determined by expressions (5).

The rubber-metal support with forces (8) on its surface is in equilibrium. From expression (8) it follows that a uniformly distributed load is applied to the upper metal plate and its value increases gradually.

Let us give a solution to the problem by the method [4]:

$$\sigma_{ij}=\delta_{i2}\delta_{j2}cx_3,\ x_i\in V \tag{9}$$

Move functions can also be defined from the expression:

$$u_i=\frac{1}{E}\int_l c\Big(-\nu\delta_{ik}x_3+(1+\nu)\delta_{i2}\delta_{k2}x_3+$$

$$\left(x_j-y_j\right)\left(-\nu\left(\delta_{ki}\delta_{3j}-\delta_{kj}\delta_{3i}\right)+(1+\nu)\delta_{k2}\left(\delta_{i2}\delta_{3j}-\delta_{j2}\delta_{3i}\right)\right)\Big)dy_k,\ x_i\in V\,.$$

Integrating this expression, and we can find:

$$u_i(x)=-c\Big(\delta_{i1}\nu x_3\left(x_1-x_1^0\right)-\delta_{i2}x_3\left(x_2-x_2^0\right)+\delta_{i3}\Big(x_2^2+\nu\left(x_3^2-x_1^2\right)-x_2^0\left(2x_2-x_2^0\right)-$$

$$-\nu\left(\left(x_3^0\right)^2-x_1^0\left(2x_1-x_1^0\right)\right)\Big)/2\Big)/E,\quad x_i\in V\quad , \tag{10}$$

where x_i^0 is any fixed point of the region ***V***. Let us present the expanded form of functions (10):

$$u_1(x)=-\frac{c\,\nu\,x_3\left(x_1-x_1^0\right)}{E},\ u_2(x)=\frac{c\ x_3\left(x_2-x_2^0\right)}{E},\ x_i\in V,$$

$$u_3(x)=-c\left(\left(x_2^2+\nu\left(x_3^2-x_1^2\right)-x_2^0\left(2x_2-x_2^0\right)-\nu\left(\left(x_3^0\right)^2-x_1^0\left(2x_1-x_1^0\right)\right)\right)/2E\right),\ x_i\in V\,.$$

We will now consider three load cases: $c=0,\ c=0.3,\ c=0.6$. There are no external forces on the surface ***S***, the body occupies the region ***V*** (5) and is in a state of equilibrium. Figure 2 shows: on the left without an applied load and on the right with an applied load. As you can take the coordinates of any point in the area (5). In what follows, we will take for the initial coordinate: $x_1^0=0.05,\ x_2^0=0.05,\ x_3^0=0$.

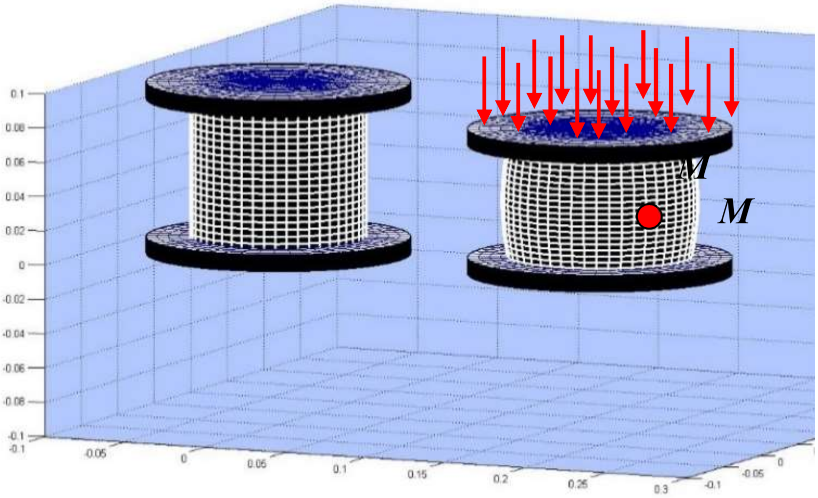

Fig. 2. *Left:* the state of equilibrium of the body, *right:* the state of equilibrium support under the distributed load of c=0.3.

We will consider the point ***M***, located inside the rubber layer, which moves under the action of compressive forces. In this article, we will define the displacement of point ***M*** under the action of a distributed load on the upper metal plate. This point is in the middle of the rubber layer in height and has the greatest displacement values when the rubber layer bulges. The modulus of elasticity for rubber grade SNK 3826 is taken as ***E***=12MPa, Poisson's ratio ***ν***=0.4995.

For the proposed mathematical model, when c=0.3 (external load is 3.82 MPa), a special program code was written in the Matlab software package. In this system, the components of the displacement of the point ***M*** in the rubber layer, were calculated (in meters).

$$u_{ij} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.0084 \end{pmatrix}.$$

The strain tensor calculated in Matlab is determined by the formulas $\varepsilon_{ij} = 0.5\left(u_{i,j} + u_{j,i}\right)$. Strain tensor in expanded form:

$$\varepsilon_{ij} := c \cdot \begin{bmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x_1}u_1(x_1,x_2,x_3) + \frac{\partial}{\partial x_1}u_1(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_2}u_1(x_1,x_2,x_3) + \frac{\partial}{\partial x_1}u_2(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_3}u_1(x_1,x_2,x_3) + \frac{\partial}{\partial x_1}u_3(x_1,x_2,x_3)\right) \\ \frac{1}{2}\left(\frac{\partial}{\partial x_1}u_2(x_1,x_2,x_3) + \frac{\partial}{\partial x_2}u_1(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_2}u_2(x_1,x_2,x_3) + \frac{\partial}{\partial x_2}u_2(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_3}u_2(x_1,x_2,x_3) + \frac{\partial}{\partial x_2}u_3(x_1,x_2,x_3)\right) \\ \frac{1}{2}\left(\frac{\partial}{\partial x_1}u_3(x_1,x_2,x_3) + \frac{\partial}{\partial x_3}u_1(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_2}u_3(x_1,x_2,x_3) + \frac{\partial}{\partial x_3}u_2(x_1,x_2,x_3)\right) & \frac{1}{2}\left(\frac{\partial}{\partial x_3}u_3(x_1,x_2,x_3) + \frac{\partial}{\partial x_3}u_3(x_1,x_2,x_3)\right) \end{bmatrix}$$

Strain tensor components for point ***M***:

$$\varepsilon_{ij} = \begin{pmatrix} 0.064 & 0 & 0 \\ 0 & 0.064 & 0 \\ 0 & 0 & 0.09 \end{pmatrix}.$$

To calculate the stress tensor at point ***M***, we use the generalized Hooke's law:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

For the cylinder we accept, $\varepsilon_x = \varepsilon_y$, $\sigma_x = \sigma_y$, $\nu = 0.4995$. As a result, we get the stress components, in MPa:

$$\sigma_{ij} = \begin{pmatrix} 1.495 & 0 & 0 \\ 0 & 1.495 & 0 \\ 0 & 0 & 1.504 \end{pmatrix}$$

Let us increase the value of the load and consider the case when c=0.6 (the load is 7.64 MPa). Under the action of the applied forces, the body occupies the same area in space.

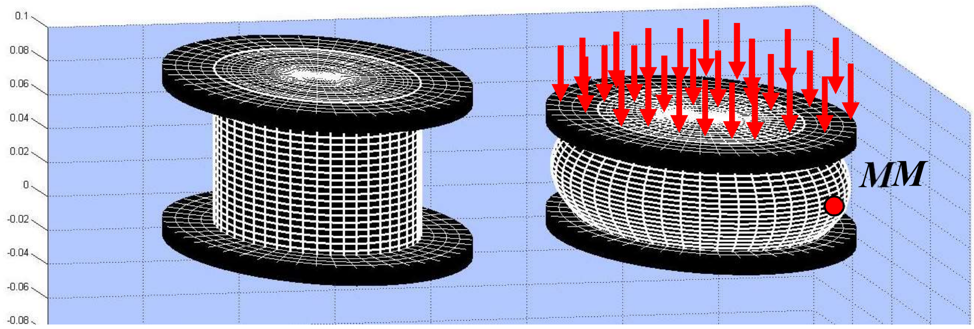

Fig. 3. RMS#2 under the influence of compressive forces c=0.6.

Displacement field for point M, in meters:

$$u_{ij} = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.0168 \end{pmatrix}$$

Strain tensor components for point ***M***:

$$\varepsilon_{ij} = \begin{pmatrix} 0.255 & 0 & 0 \\ 0 & 0.255 & 0 \\ 0 & 0 & 0.36 \end{pmatrix}$$

Stress tensor for point ***M***, in MPa:

$$\sigma_{ij} = \begin{pmatrix} 5.981 & 0 & 0 \\ 0 & 5.981 & 0 \\ 0 & 0 & 6.018 \end{pmatrix}$$

This problem (1), (2), (3) can also be represented by the Navier equations:

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} = 0, \; x_i \in V. \tag{11}$$

The boundary conditions for these equations can be written in three forms:

• The displacements on the surface S are given, which are determined by the function (10).

• Specified external forces on the surface S, determined by expression (8).

• Defined on the displacement faces, defined by expression (10), into which the following coordinate values must be substituted in turn $x_1 = -0.05$, $x_1 = 0.05$, $x_2 = -0.05$, $x_2 = 0.05$ and on the rest of the surface:

$$\sigma_{ij}\left(x_1, x_2, 0\right) = -\delta_{i2} c x_3, \; \sigma_{ij}\left(x_1, x_2, 0.1\right) = \delta_{i2} c x_3 \qquad (13)$$

There are many works of mechanics where the Cauchy and Almansi strain measures are generally accepted solutions. When the points of the body move relative to each other, the relations between the stress and strain tensors are the laws governing such movement. Deformations can be expressed in terms of displacement gradients given this condition. The solution of the problem will become more complicated if these expressions have nonlinearity. From a mathematical point of view, for scientists of mechanics, the deformation of the body, or rather the process of deformation of the body and the description of this process has always been an urgent task.

The description of the deformation process of such elastic materials as rubber or rubber-like elements using the new method is very interesting. The capabilities of the Cauchy tensor allows you to use the fundamental foundations of the theory of elasticity with the use of new approaches to study the behavior of a material under the influence of external loads.

Conclusion

In this work, we present the results of applying a mathematical model based on the analytical method, which allows to us describing the deformed and stressed states of rubber-metal elements under the influence of compressive forces. This direction refers to topical problems of mechanics and is closely related to the solution of various engineering and technical problems. The model determines the stress state of the RMS and can be successfully applied to the calculation of small, final, and large deformations of rubber-metal supports. The proposed mathematical model develops in relation to rubber-like materials, the analytical method for solving static boundary value problems of the theory of elasticity, proposed by Professor Turatbek Duishenaliev, is very convenient when using modern computational programs [1,4,13]. New results are obtained based on the application of fundamental principles and laws of continuum mechanics.

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